

HOLONOMY GROUPS OF COMPLETE FLAT PSEUDO-RIEMANNIAN HOMOGENEOUS SPACES

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ABSTRACT. We show that a complete flat pseudo-Riemannian homogeneous manifold with non-abelian linear holonomy is of dimension ≥ 14 . Due to an example constructed in a previous article [2], this is a sharp bound. Also, we give a structure theory for the fundamental groups of complete flat pseudo-Riemannian manifolds in dimensions ≤ 6 . Finally, we observe that every finitely generated torsion-free 2-step nilpotent group can be realized as the fundamental group of a complete flat pseudo-Riemannian manifold with abelian linear holonomy.

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1. INTRODUCTION

The study of pseudo-Riemannian homogeneous space forms was pioneered by Joseph A. Wolf in the 1960s. In the flat case, he proved that the fundamental group Γ of such a space M is 2-step nilpotent. For M with abelian linear holonomy group he derived a representation by unipotent affine transformations [9]. The linear holonomy group $\text{Hol}(\Gamma)$ of M is the group consisting of the linear parts of Γ . Wolf further proved that for $\dim M \leq 4$ and Lorentz signatures, Γ is a group of pure translations, and that Γ is free abelian for signatures $(n - 2, 2)$. It was unclear whether or not non-abelian Γ could exist for other signatures, until Oliver Baues gave a first example in [1] of a compact flat pseudo-Riemannian homogeneous

space with signature $(3, 3)$ having non-abelian fundamental group and abelian linear holonomy group.

In an article [2] by Oliver Baues and the author, Wolf's unipotent representations for fundamental groups with abelian $\text{Hol}(\Gamma)$ were generalized for groups with non-abelian linear holonomy. Also, it was shown that a (possibly incomplete) flat pseudo-Riemannian homogeneous manifold M with non-abelian linear holonomy group is of dimension $\dim M \geq 8$. In chapter 2 we review the main results about the algebraic structure of the fundamental and holonomy groups of such M .

It was asserted in [2] that if M is (geodesically) complete, then $\dim M \geq 14$ holds. This assertion is proved in chapter 3 of the present article. More precisely, we prove the following:

Theorem. *If M is a complete flat homogeneous pseudo-Riemannian manifold such that its fundamental group Γ has non-abelian linear holonomy group, then*

$$\dim M \geq 14$$

and the signature $(n - s, s)$ of M satisfies $n - s \geq s \geq 7$.

This estimate is sharp by an example given in [2], which is repeated in Example 3.12 for the reader's convenience.

In chapter 4 we give a complete description of the fundamental groups of flat pseudo-Riemannian homogeneous spaces up to dimension 6. Although non-abelian fundamental groups may occur in dimension 6, their holonomy groups are abelian as a consequence of the dimension bound in the above theorem.

Further, we will see in chapter 5 how any finitely generated torsion-free 2-step nilpotent group can be realized as the fundamental group of a complete flat pseudo-Riemannian homogeneous manifold with abelian holonomy:

Theorem. *Let Γ be a finitely generated torsion-free 2-step nilpotent group of rank n . Then there exists a faithful representation $\varrho : \Gamma \rightarrow \text{Iso}(\mathbb{R}_n^{2n})$ such that $M = \mathbb{R}_n^{2n} / \varrho(\Gamma)$ a complete flat pseudo-Riemannian homogeneous manifold M of signature (n, n) with abelian linear holonomy group.*

2. PRELIMINARIES

Let \mathbb{R}_s^n be the space \mathbb{R}^n endowed with a non-degenerate symmetric bilinear form of signature $(n - s, s)$ and $\text{Iso}(\mathbb{R}_s^n)$ its group of isometries. We assume $n - s \geq s$ throughout. The number s is called the *Witt index*. For a vector space V endowed with a non-degenerate symmetric bilinear form let $\text{wi}(V)$ denote its Witt index. Affine maps of \mathbb{R}^n are written as $\gamma = (I + A, v)$, where $I + A$ is the linear part (I the identity matrix), and v the translation part. Let $\text{im } A$ denote the image of A .

Let M denote a complete flat pseudo-Riemannian homogeneous manifold. Then M is of the form $M = \mathbb{R}_s^n / \Gamma$ with fundamental group $\Gamma \subset \text{Iso}(\mathbb{R}_s^n)$. Homogeneity is determined by the action of the centralizer $Z_{\text{Iso}(\mathbb{R}_s^n)}(\Gamma)$ of Γ in $\text{Iso}(\mathbb{R}_s^n)$ (see [8, Theorem 2.4.17]):

Theorem 2.1. *Let $p : \tilde{M} \rightarrow M$ be the universal pseudo-Riemannian covering of M and let Γ be the group of deck transformations. Then M is homogeneous if and only if $Z_{\text{Iso}(\mathbb{R}_s^n)}(\Gamma)$ acts transitively on \tilde{M} .*

This condition further implies that Γ acts without fixed points on \mathbb{R}_s^n . This constraint on Γ is the main difference to the more general case where M is not required to be geodesically complete.

Now assume $\Gamma \subset \text{Iso}(\mathbb{R}_s^n)$ has transitive centralizer in $\text{Iso}(\mathbb{R}_s^n)$. We sum up some properties of Γ for later reference (these are originally due to [9], see also [5], [8] for reference).

Lemma 2.2. Γ consists of affine transformations $\gamma = (I + A, v)$, where $A^2 = 0$, $v \perp \text{im } A$ and $\text{im } A$ is totally isotropic.

Lemma 2.3. For $\gamma_i = (I + A_i, v_i) \in \Gamma$, $i = 1, 2, 3$, we have $A_1 A_2 v_1 = 0 = A_2 A_1 v_2$, $A_1 A_2 A_3 = 0$ and $[\gamma_1, \gamma_2] = (I + 2A_1 A_2, 2A_1 v_2)$.

Lemma 2.4. If $\gamma = (I + A, v) \in \Gamma$, then $\langle Ax, y \rangle = -\langle x, Ay \rangle$, $\text{im } A = (\ker A)^\perp$, $\ker A = (\text{im } A)^\perp$ and $Av = 0$.

Theorem 2.5. Γ is 2-step nilpotent (meaning $[\Gamma, [\Gamma, \Gamma]] = \{\text{id}\}$).

For $\gamma = (I + A, v) \in \Gamma$, set $\text{Hol}(\gamma) = I + A$ (the linear component of γ). We write $A = \log(\text{Hol}(\gamma))$.

Definition 2.6. The linear holonomy group of Γ is $\text{Hol}(\Gamma) = \{\text{Hol}(\gamma) \mid \gamma \in \Gamma\}$.

Let $x \in M$ and $\gamma \in \pi_1(M, x)$ be a loop. Then $\text{Hol}(\gamma)$ corresponds to the parallel transport $\tau_x(\gamma) : T_x M \rightarrow T_x M$ in a natural way, see [8, Lemma 3.4.4]. This justifies the naming.

Proposition 2.7. The following are equivalent:

- (1) $\text{Hol}(\Gamma)$ is abelian.
- (2) If $(I + A_1, v_1), (I + A_2, v_2) \in \Gamma$, then $A_1 A_2 = 0$.
- (3) The space $U_\Gamma = \sum_{\gamma \in \Gamma} \text{im } A$ is totally isotropic.

Those Γ with possibly non-abelian $\text{Hol}(\Gamma)$ were studied in [2]: If $\text{Hol}(\Gamma)$ is not abelian, the space U_Γ is not totally isotropic. So we replace U_Γ by the totally isotropic subspace

$$(2.1) \quad U_0 = U_\Gamma \cap U_\Gamma^\perp = \sum_{\gamma \in \Gamma} \text{im } A \cap \bigcap_{\gamma \in \Gamma} \ker A.$$

We can find a Witt basis for \mathbb{R}_s^n with respect to U_0 , that is a basis with the following properties: If $k = \dim U_0$, there exists a basis for \mathbb{R}_s^n ,

$$(2.2) \quad \{u_1, \dots, u_k, w_1, \dots, w_{n-2k}, u_1^*, \dots, u_k^*\},$$

such that $\{u_1, \dots, u_k\}$ is a basis of U_0 , $\{w_1, \dots, w_{n-2k}\}$ is a basis of a non-degenerate subspace W such that $U_0^\perp = U_0 \oplus W$, and $\{u_1^*, \dots, u_k^*\}$ is a basis of a space U_0^* such that $\langle u_i, u_j^* \rangle = \delta_{ij}$ (then U_0^* is called a dual space for U_0). Then

$$(2.3) \quad \mathbb{R}_s^n = U_0 \oplus W \oplus U_0^*$$

is called a Witt decomposition of \mathbb{R}_s^n . Let \tilde{I} denote the signature matrix representing the restriction of $\langle \cdot, \cdot \rangle$ to W with respect to the chosen basis of W .

In [2, Theorem 4.4] we derived the following representation for Γ :

Theorem 2.8. Let $\gamma = (I + A, v) \in \Gamma$ and fix a Witt basis with respect to U_0 . Then the matrix representation of A in this basis is

$$(2.4) \quad A = \begin{pmatrix} 0 & -B^\top \tilde{I} & C \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix},$$

with $B \in \mathbb{R}^{(n-2k) \times k}$ and $C \in \mathfrak{so}_k$ (where $k = \dim U_0$). The columns of B are isotropic and mutually orthogonal with respect to \tilde{I} .

3. THE DIMENSION BOUND FOR COMPLETE MANIFOLDS

In this section we derive further properties of the matrix representation in (2.4).

3.1. Properties of the Matrix Representation. We fix a Witt basis for U_0 as in the previous section. Let $\gamma_i \in \Gamma$ with $\gamma_i = (I + A_i, v_i)$, $i = 1, 2$. Then B_i and C_i refer to the respective matrix blocks of A_i in (2.4). Set $[\gamma_1, \gamma_2] = \gamma_3 = (I + A_3, v_3)$.

Lemma 3.1. *We have $v_3 = 2A_1v_2 = -2A_2v_1 \in U_0$. Further, if $\gamma_3 \neq I$ and Γ acts freely, then $v_3 \neq 0$.*

Proof. By Lemma 2.3, $v_3 = 2A_1v_2 = -2A_2v_1 \in \text{im } A_1$. Because Γ is 2-step nilpotent, γ_3 is central. Again by Lemma 2.3, $v_3 \in \bigcap_{\gamma \in \Gamma} \ker A$. Hence $v_3 \in U_0$.

If Γ acts freely and $\gamma_3 \neq I$, then $v_3 \neq 0$ because otherwise 0 would be a fixed point for γ_3 . \square

Lemma 3.2. *If u_1^*, u_2^* denote the respective U_0^* -components of the translation parts v_1, v_2 , then $u_1^*, u_2^* \in \ker B_1 \cap \ker B_2$.*

Proof. Let $v_3 = u_3 + w_3 + u_3^*$ be the Witt decomposition of v_3 . By Lemma 3.1, $w_3 = 0, u_3^* = 0$. Writing out the equation $v_3 = A_1v_2 = -A_2v_1$ with (2.4) it follows that $B_1u_2^* = 0 = B_2u_1^*$. By Lemma 2.4, $B_1u_1^* = 0 = B_2u_2^*$. \square

The following rules were already used in [2, Theorem 5.1] to derive the general dimension bound for (possibly incomplete) flat pseudo-Riemannian homogeneous manifolds:

- (1) *Isotropy rule:* The columns of B_i are isotropic and mutually orthogonal with respect to \tilde{I} (Theorem 2.8).
- (2) *Crossover rule:* Given A_1 and A_2 , let b_2^i be a column of B_2 and b_1^k a column of B_1 . Then $\langle b_1^k, b_2^i \rangle = -\langle b_1^i, b_2^k \rangle$. In particular, $\langle b_1^k, b_2^k \rangle = 0$, and $\langle b_1^i, b_1^i \rangle = 0$. If $\langle b_1^i, b_2^k \rangle \neq 0$ then $b_1^k, b_1^i, b_2^k, b_2^i$ are linearly independent. (The product of A_1A_2 contains $-B_1^T \tilde{I} B_2$ as the skew-symmetric upper right block, so its entries are the values $-\langle b_1^k, b_2^i \rangle$.)
- (3) *Duality rule:* Assume A_1 is not central (that is $A_1A_2 \neq 0$ for some A_2). Then B_2 contains a column b_2^i and B_1 a column b_1^j such that $\langle b_1^j, b_2^i \rangle \neq 0$.

Lemma 3.3. *Assume $A_1A_2 \neq 0$ and that the columns b_1^i in B_1 and b_2^j in B_2 satisfy $\langle b_1^i, b_2^j \rangle \neq 0$. The subspace W in (2.3) has a Witt decomposition*

$$(3.1) \quad W = W_{ij} \oplus W' \oplus W_{ij}^*,$$

where $W_{ij} = \mathbb{R}b_1^i \oplus \mathbb{R}b_1^j$, $W_{ij}^* = \mathbb{R}b_2^i \oplus \mathbb{R}b_2^j$, $W' \perp W_{ij}$, $W' \perp W_{ij}^*$, and $\langle \cdot, \cdot \rangle$ is non-degenerate on W' . Furthermore,

$$(3.2) \quad \text{wi}(W) \geq \text{rk } B_1 \geq 2 \quad \text{and} \quad \dim W \geq 2 \text{rk } B_1 \geq 4.$$

Proof. $\mathbb{R}b_1^i \oplus \mathbb{R}b_1^j$ is totally isotropic because $\text{im } B_1$ is. By the crossover rule, $\{b_2^j, b_2^i\}$ is a dual basis to $\{b_1^i, b_1^j\}$ (after scaling, if necessary).

W contains $\text{im } B_1$ as a totally isotropic subspace, so it also contains a dual space. Hence $\text{wi}(W) \geq \text{rk } B_1 \geq \dim W_{ij} \geq 2$ and $\dim W \geq 2 \text{rk } B_1 \geq 2 \dim W_{ij} = 4$. \square

3.2. Criteria for Fixed Points. In this subsection, assume the centralizer of $\Gamma \subset \text{Iso}(\mathbb{R}_s^n)$ has an open orbit in \mathbb{R}_s^n , but does not necessarily act transitively.

Remark 3.4. If the centralizer does act transitively on \mathbb{R}_s^n , then Γ acts freely: Assume $\gamma.p = p$ for some $\gamma \in \Gamma$, $p \in \mathbb{R}_s^n$. For every $q \in \mathbb{R}_s^n$ there is $z \in Z_{\text{Iso}(\mathbb{R}_s^n)}(\Gamma)$ such that $z.p = q$. So $\gamma.q = \gamma.(z.p) = z.(\gamma.p) = z.p = q$ for all $q \in \mathbb{R}_s^n$. Hence $\gamma = I$.

We will deduce some criteria for Γ to have a fixed point, which allows us to exclude such groups Γ as fundamental groups for *complete* flat pseudo-Riemannian homogeneous manifolds.

Let $\Gamma, U_0, \gamma_1, \gamma_2, \gamma_3 = [\gamma_1, \gamma_2], A_i, B_i, C_i$ be as in the previous sections. For any $v \in \mathbb{R}_s^n$ let $v = u + w + u^*$ denote the Witt decomposition with respect to U_0 . From (2.4) we get the following two coordinate expressions which we use repeatedly:

$$(3.3) \quad A_1 A_2 = \begin{pmatrix} 0 & -B_1^\top \tilde{I} & C_1 \\ 0 & 0 & B_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -B_2^\top \tilde{I} & C_2 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -B_1^\top \tilde{I} B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(note that $A_3 = 2A_1 A_2$ as a consequence of Lemma 2.3), and for $v \in \mathbb{R}_s^n$

$$(3.4) \quad A_i v = \begin{pmatrix} 0 & -B_i^\top \tilde{I} & C_i \\ 0 & 0 & B_i \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ u^* \end{pmatrix} = \begin{pmatrix} -B_i^\top \tilde{I} w + C_i u^* \\ B_i u^* \\ 0 \end{pmatrix}.$$

In the following we assume that the linear parts of γ_1, γ_2 do not commute, that is $A_1 A_2 \neq 0$. In particular, $A_1 v_2 \neq 0$.

Lemma 3.5. *If $u_3 \in \text{im } B_1^\top \tilde{I} B_2$, then γ_3 has a fixed point.*

Proof. We have $C_3 = -B_1^\top \tilde{I} B_2$ by Lemma 2.3 and (2.4). By Lemma 3.1, $v_3 = u_3 \in U_0$. If there exists $u^* \in U_0^*$ such that $C_3 u^* = u_3$, then $\gamma_3.(-u^*) = (I + A_3, v_3).(-u^*) = -u^* - C_3 u^* + u_3 = -u^*$. So $-u^*$ is fixed by γ_3 . \square

Lemma 3.6. *If $\text{rk } B_1^\top \tilde{I} B_2 = \text{rk } B_1$ and the Γ -action is free, then $u_1^* \neq 0, u_2^* \neq 0$.*

Proof. From (3.4) we get $u_3 = -B_1^\top \tilde{I} w_2 + C_1 u_2^*$. Also, $\text{im } B_1^\top \tilde{I} B_2 \subset \text{im } B_1^\top$. But by our rank assumption, $\text{im } B_1^\top \tilde{I} B_2 = \text{im } B_1^\top$.

So, if $u_2^* = 0$, then $u_3 \in \text{im } B_1^\top = \text{im } B_1^\top \tilde{I} B_2$, which implies the existence of a fixed point by Lemma 3.5. So $u_2^* \neq 0$ if the action is free. Using $v_3 = A_1 v_2 = -A_2 v_1$, we can conclude $u_1^* \neq 0$ in a similar manner. \square

Corollary 3.7. *If $\dim U_0 = 2$, then the Γ -action is not free.*

Proof. By Lemma 3.3, $2 \leq \text{rk } B_1 \leq \dim U_0 = 2$, so B_1 is of full rank. Now $A_1 v_1 = 0$ and (3.4) imply $u_1^* = 0$, so by Lemma 3.6, the Γ -action is not free. \square

Lemma 3.8. *If $\dim U_0 = 3$ and $\dim(\text{im } B_1 + \text{im } B_2) \leq 5$, then γ_3 has a fixed point.*

Proof. By Lemma 3.3, $\text{rk } B_1, \text{rk } B_2 \geq 2$. We distinguish two cases:

- (i) Assume $\text{rk } B_1 = 2$ (or $\text{rk } B_2 = 2$). Because $C_3 = -B_1^\top \tilde{I} B_2 \neq 0$ is skew, it is also of rank 2. Then $\text{im } B_1^\top \tilde{I} B_2 = \text{im } B_1^\top$. $\ker B_1$ is a 1-dimensional subspace due to $\dim U_0^* = 3$. Because $u_1^*, u_2^* \in \ker B_1$, we have $u_1^* = \lambda u_2^*$ for some number $\lambda \neq 0$.

From (3.4) and $A_1 v_1 = 0$ we get

$$\begin{aligned}\lambda u_3 &= -B_1^\top \tilde{I} \lambda w_2 + C_1 \lambda u_2^* = -B_1^\top \tilde{I} \lambda w_2 + C_1 u_1^*, \\ 0 &= -B_1^\top \tilde{I} w_1 + C_1 u_1^*.\end{aligned}$$

So $\lambda u_3 = \lambda u_3 - 0 = B_1^\top \tilde{I}(w_1 - \lambda w_2)$. In other words, $u_3 \in \text{im } B_1^\top = \text{im } B_1^\top \tilde{I} B_2$, and γ_3 has a fixed point by Lemma 3.5.

- (ii) Assume $\text{rk } B_1 = \text{rk } B_2 = 3$. As $[A_1, A_2] \neq 0$, the duality rule and the crossover rule imply the existence of a pair of columns b_1^i, b_1^j in B_1 and a pair of columns b_2^i, b_2^j in B_2 such that $\alpha = \langle b_1^i, b_2^j \rangle = -\langle b_1^j, b_2^i \rangle \neq 0$. For simplicity say $i = 1, j = 2$. As $\text{rk } B_1 = 3$, the column b_1^3 is linearly independent of b_1^1, b_1^2 , and these columns span the totally isotropic subspace $\text{im } B_1$ of W .

- Assume $b_2^3 \in \text{im } B_1$ (or $b_1^3 \in \text{im } B_2$). Then b_2^3 is a multiple of b_1^3 : In fact, let $b_2^3 = \lambda_1 b_1^1 + \lambda_2 b_1^2 + \lambda_3 b_1^3$. Then $\langle b_2^3, b_1^1 \rangle = 0$ because $\text{im } B_1$ is totally isotropic. Since $\text{im } B_2$ is totally isotropic and by the crossover rule,

$$\begin{aligned}0 &= \langle b_2^3, b_2^1 \rangle = \lambda_1 \langle b_1^1, b_2^1 \rangle + \lambda_2 \langle b_1^2, b_2^1 \rangle + \lambda_3 \langle b_1^3, b_2^1 \rangle \\ &= \lambda_2 \alpha - \lambda_3 \langle b_2^3, b_1^1 \rangle = \lambda_2 \alpha.\end{aligned}$$

Because $\alpha \neq 0$, this implies $\lambda_2 = 0$ and in the same way $\lambda_1 = 0$. So $b_2^3 = \lambda_3 b_1^3$. Now $b_1^3 \perp b_1^i, b_2^j$ for all i, j . We have $u_2^* = 0$ because $B_2 u_2^* = 0$ and B_2 is of maximal rank. Then $\langle b_1^3, w_2 \rangle = \langle b_2^3, w_2 \rangle = 0$, because $0 = B_2^\top \tilde{I} w_2 + C_2 u_2^* = B_2^\top \tilde{I} w_2$. Hence C_3 and u_3 take the form

$$C_3 = -B_1^\top \tilde{I} B_2 = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, u_3 = -B_1^\top \tilde{I} w_2 = \begin{pmatrix} -\langle b_1^1, w_2 \rangle \\ -\langle b_1^2, w_2 \rangle \\ 0 \end{pmatrix}.$$

It follows that $u_3 \in \text{im } C_3$, so in this case γ_3 has a fixed point by Lemma 3.5.

- Assume $b_2^3 \notin \text{im } B_1$ and $b_1^3 \notin \text{im } B_2$. This means b_2^3 and b_1^3 are linearly independent. If $b_2^3 \perp \text{im } B_1$, then $b_1^3 \perp \text{im } B_2$ by the crossover rule. With respect to the Witt decomposition $W = W_{12} \oplus W' \oplus W_{12}^*$ (Lemma 3.3), this means b_1^3, b_2^3 span a 2-dimensional subspace of $(W_{12} \oplus W_{12}^*)^\perp = W'$. But then $\dim(\text{im } B_1 + \text{im } B_2) = 6$, contradicting the lemma's assumption that this dimension should be ≤ 5 .

So $b_2^3 \not\perp \text{im } B_1$ and $b_1^3 \not\perp \text{im } B_2$ hold. Because further $b_1^3 \perp \text{im } B_1$, $b_2^3 \perp \text{im } B_2$ and $\dim(\text{im } B_1 + \text{im } B_2) \leq 5$, there exists a $b \in W'$ (with W' from the Witt decomposition above) such that

$$\begin{aligned}b_1^3 &= \lambda_1 b_1^1 + \lambda_2 b_1^2 + \lambda_3 b, \\ b_2^3 &= \mu_1 b_2^1 + \mu_2 b_2^2 + \mu_3 b.\end{aligned}$$

Because B_1, B_2 are of maximal rank, we have $u_1^* = 0 = u_2^*$ as a consequence of $A_i v_i = 0$. Then

$$0 = B_2^\top \tilde{I} w_2 = \begin{pmatrix} \langle b_2^1, w_2 \rangle \\ \langle b_2^2, w_2 \rangle \\ \langle b_2^3, w_2 \rangle \end{pmatrix},$$

and this implies $\langle b, w_2 \rangle = 0$. Put $\xi = \langle b_1^1, w_2 \rangle$, $\eta = \langle b_1^2, w_2 \rangle$. Then

$$u_3 = -B_1^\top \tilde{I} w_2 = - \begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix}$$

and (recall $\alpha = \langle b_1^1, b_2^2 \rangle = -\langle b_1^2, b_2^1 \rangle$)

$$C_3 = B_2^\top \tilde{I} B_1 = \begin{pmatrix} 0 & -\alpha & -\lambda_2 \alpha \\ \alpha & 0 & \lambda_1 \alpha \\ \lambda_2 \alpha & -\lambda_1 \alpha & 0 \end{pmatrix}.$$

So

$$C_3 \cdot \frac{1}{\alpha} \begin{pmatrix} -\eta \\ \xi \\ 0 \end{pmatrix} = - \begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix} = u_3.$$

By Lemma 3.5, γ_3 has a fixed point.

So in any case γ_3 has a fixed point. \square

Lemma 3.9. *If $\dim U_0 = 4$ and $\text{rk } B_1^\top \tilde{I} B_2 = \text{rk } B_1 = \text{rk } B_2$, then γ_3 has a fixed point.*

Proof. By assumption

$$\text{im } B_1^\top \tilde{I} B_2 = \text{im } B_1^\top = \text{im } B_2^\top.$$

First, assume $u_1^* = \lambda u_2^*$ for some number $\lambda \neq 0$. Writing out $A_1 v_2 = v_3$ and $A_1 v_1 = 0$ with (3.4), we get

$$\begin{aligned} \lambda u_3 &= -B_1^\top \tilde{I} \lambda w_2 + C_1 \lambda u_2^* = -B_1^\top \tilde{I} \lambda w_2 + C_1 u_1^*, \\ 0 &= -B_1^\top \tilde{I} w_1 + C_1 u_1^*. \end{aligned}$$

So

$$\lambda u_3 = \lambda u_3 - 0 = B_1^\top \tilde{I} (w_1 - \lambda w_2).$$

In other words, $u_3 \in \text{im } B_1^\top = \text{im } B_1^\top \tilde{I} B_2$, and γ_3 has a fixed point by Lemma 3.5.

Now, assume u_1^* and u_2^* are linearly independent. Lemma 3.2 can be reformulated as

$$\text{im } B_1^\top = \text{im } B_2^\top \subseteq \ker u_1^{*\top} \cap \ker u_2^{*\top}.$$

$\ker u_1^{*\top}$, $\ker u_2^{*\top}$ are 3-dimensional subspaces of the 4-dimensional space U_0^* , and their intersection is of dimension 2 (because u_1^*, u_2^* are linearly independent). By Lemma 3.3, $\text{rk } B_1 \geq 2$, so it follows that

$$\text{im } B_1^\top = \text{im } B_2^\top = \ker u_1^{*\top} \cap \ker u_2^{*\top}.$$

With $A_1 v_1 = 0$ and (3.4) we conclude $C_1 u_1^* = b$ for some $b \in \text{im } B_1^\top$. Thus, by the skew-symmetry of C_1 ,

$$0 = (u_2^{*\top} C_1 u_1^*)^\top = -u_1^{*\top} C_1 u_2^*.$$

So $C_1 u_2^* \in \ker u_1^{*\top}$. In the same way $C_2 u_1^* \in \ker u_2^{*\top}$. But $u_3 = C_1 u_2^* + b_1 = -C_2 u_1^* + b_2$ for some $b_1, b_2 \in \text{im } B_1^\top$. Hence

$$\begin{aligned} u_1^{*\top} u_3 &= \underbrace{u_1^{*\top} C_1 u_2^*}_{=0} + \underbrace{u_1^{*\top} b_1}_{=0} = 0, \\ u_2^{*\top} u_3 &= -\underbrace{u_2^{*\top} C_2 u_1^*}_{=0} + \underbrace{u_2^{*\top} b_2}_{=0} = 0. \end{aligned}$$

So $u_3 \in \ker u_1^{*\top} \cap \ker u_2^{*\top} = \operatorname{im} B_1^\top = \operatorname{im} B_1^\top \tilde{I} B_2$. With Lemma 3.5 we conclude that there exists a fixed point for γ_3 . \square

3.3. The Dimension Bound. Let $\Gamma, \gamma_1, \gamma_2, \gamma_3 = [\gamma_1, \gamma_2]$ be as in the previous subsection, let $\mathbb{R}_s^n = U_0 \oplus W \oplus U_0^*$ be the Witt decomposition (2.3), and let A_i, B_i, C_i refer to the matrix representation (2.4) of γ_i . We will assume that the linear parts A_1, A_2 of γ_1, γ_2 do not commute, that is, $\operatorname{Hol}(\Gamma)$ is not abelian.

Theorem 3.10. *Let $\Gamma \subset \operatorname{Iso}(\mathbb{R}_s^n)$ and assume the centralizer $Z_{\operatorname{Iso}(\mathbb{R}_s^n)}(\Gamma)$ acts transitively on \mathbb{R}_s^n . If $\operatorname{Hol}(\Gamma)$ is non-abelian, then*

$$s \geq 7 \quad \text{and} \quad n \geq 14.$$

As Example 3.12 shows, this is a sharp lower bound.

Proof. We will show $s \geq 7$, then it follows immediately from $n - s \geq s$ that

$$n \geq 2s \geq 14.$$

If the centralizer is transitive, then Γ acts freely. From Corollary 3.7 we know that $\dim U_0 \geq 3$. By Lemma 3.3, $\operatorname{wi}(W) \geq 2$, and if $\dim U_0 \geq 5$, then

$$s = \dim U_0 + \operatorname{wi}(W) \geq 5 + 2 = 7,$$

and we are done. So let $2 < \dim U_0 < 5$.

- (i) First, let $\dim U_0 = 4$. Assume $\operatorname{rk} B_1 = \operatorname{rk} B_2 = 2$. Because $C_3 = -B_1^\top \tilde{I} B_2 \neq 0$ is skew, it is of rank 2. So $\operatorname{rk} B_1 = \operatorname{rk} B_2 = 2 = \operatorname{rk} B_1^\top \tilde{I} B_2$. By Lemma 3.9, the action of Γ is not free.

Now assume $\operatorname{rk} B_1 \geq 3$. It follows from Lemma 3.3 that $\operatorname{wi}(W) \geq 3$ and $\dim W \geq 6$, so once more

$$s = \dim U_0 + \operatorname{wi}(W) \geq 4 + 3 = 7.$$

So the theorem holds for $\dim U_0 = 4$.

- (ii) Let $\dim U_0 = 3$. If $\dim(\operatorname{im} B_1 + \operatorname{im} B_2) \leq 5$, there exists a fixed point by Lemma 3.8, so Γ would not act freely. So let $\dim(\operatorname{im} B_1 + \operatorname{im} B_2) = 6$: As $[A_1, A_2] \neq 0$, the crossover rule implies the existence of a pair of columns b_1^i, b_1^j in B_1 and a pair of columns b_2^i, b_2^j in B_2 such that $\alpha = \langle b_1^i, b_2^j \rangle = -\langle b_1^j, b_2^i \rangle \neq 0$. For simplicity say $i = 1, j = 2$. The columns b_1^1, b_1^2, b_1^3 span the totally isotropic subspace $\operatorname{im} B_1$ of W , and b_2^1, b_2^2, b_2^3 span $\operatorname{im} B_2$. We have a Witt decomposition with respect to $W_{12} = \mathbb{R}b_1^1 \oplus \mathbb{R}b_1^2$ (Lemma 3.3),

$$W = W_{12} \oplus W' \oplus W_{12}^*,$$

where $W_{12}^* = \mathbb{R}b_2^1 \oplus \mathbb{R}b_2^2$. Because $b_1^3 \perp \operatorname{im} B_1$ and $b_2^3 \perp \operatorname{im} B_2$,

$$b_1^3 = \lambda_1 b_1^1 + \lambda_2 b_1^2 + b', \quad b_2^3 = \mu_1 b_2^1 + \mu_2 b_2^2 + b'',$$

where $b', b'' \in W'$ are linearly independent because $\dim(\operatorname{im} B_1 + \operatorname{im} B_2) = 6$. From $0 = \langle b_1^3, b_1^3 \rangle$ it follows that $\langle b', b' \rangle = 0$, and similarly $\langle b'', b'' \rangle = 0$. The crossover rule then implies

$$\begin{aligned} \lambda_1 \langle b_2^2, b_1^1 \rangle &= \langle b_2^2, b_1^3 \rangle = -\langle b_2^3, b_1^2 \rangle = -\mu_1 \langle b_2^1, b_1^2 \rangle = \mu_1 \langle b_2^2, b_1^1 \rangle, \\ \lambda_2 \langle b_2^1, b_1^2 \rangle &= \langle b_2^1, b_1^3 \rangle = -\langle b_2^3, b_1^1 \rangle = -\mu_2 \langle b_2^2, b_1^1 \rangle = \mu_2 \langle b_2^1, b_1^2 \rangle. \end{aligned}$$

As the inner products are $\neq 0$, it follows that $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$. Then, by the duality rule,

$$0 = \langle b_1^3, b_2^3 \rangle = \underbrace{(\lambda_1 \mu_2 - \lambda_2 \mu_1)}_{=0} \langle b_2^2, b_1^1 \rangle + \langle b', b'' \rangle = \langle b', b'' \rangle.$$

So b' and b'' span a 2-dimensional totally isotropic subspace in the non-degenerate space W' , so this subspace has a 2-dimensional dual in W' and $\dim W' \geq 4$, $\text{wi}(W') \geq 2$, follows. Hence

$$\text{wi}(W) = \dim W_{12} + \text{wi}(W') \geq 2 + 2 = 4,$$

and again

$$s = \dim U_0 + \text{wi}(W) \geq 3 + 4 = 7.$$

In any case $s \geq 7$ and $n \geq 14$. \square

Corollary 3.11. *If M is a complete flat homogeneous pseudo-Riemannian manifold such that its fundamental group Γ has non-abelian linear holonomy group $\text{Hol}(\Gamma)$, then*

$$\dim M \geq 14$$

and the signature $(n - s, s)$ of M satisfies $n - s \geq s \geq 7$.

The dimension bound in Corollary 3.11 is sharp, as the following example from [2] shows:

Example 3.12. Let $\Gamma \subset \text{Iso}(\mathbb{R}_7^{14})$ be the group generated by

$$\gamma_1 = \left(\begin{pmatrix} I_5 & -B_1^\top \tilde{I} & C_1 \\ 0 & I_4 & B_1 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^* \end{pmatrix} \right), \quad \gamma_2 = \left(\begin{pmatrix} I_5 & -B_2^\top \tilde{I} & C_2 \\ 0 & I_4 & B_2 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2^* \end{pmatrix} \right)$$

in the basis representation (2.4). Here,

$$B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $\tilde{I} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. Their commutator is

$$\gamma_3 = [\gamma_1, \gamma_2] = \left(\begin{pmatrix} I_5 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix} \right),$$

with

$$C_3 = \begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

The group Γ is isomorphic to a discrete Heisenberg group, and the linear parts of γ_1, γ_2 do not commute. In [2, Example 6.4] it was shown that Γ has transitive centralizer in $\text{Iso}(\mathbb{R}_7^{14})$ and acts properly discontinuously and freely on \mathbb{R}_7^{14} . Hence $M = \mathbb{R}_7^{14}/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold of dimension 14 with non-abelian linear holonomy.

4. LOW DIMENSIONS

In this section, we determine the structure of the fundamental groups of complete flat pseudo-Riemannian homogeneous spaces M of dimensions ≤ 6 and of those with signature $(n-2, 2)$. The signatures $(n, 0)$, $(n-1, 1)$ and $(n-2, 2)$ were already studied by Wolf [8, Corollary 3.7.13]. In particular, he derived the following:

Proposition 4.1 (Wolf). *If M is a complete homogeneous flat Riemannian or Lorentzian manifold, then the fundamental group of M is an abelian group consisting of pure translations.*

Let $\Gamma \subset \text{Iso}(\mathbb{R}_s^n)$ denote the fundamental group of M and $G \subset \text{Iso}(\mathbb{R}_s^n)$ its real Zariski closure with Lie algebra \mathfrak{g} . Let U_Γ be as in Proposition 2.7.

We start by collecting some general facts about Γ and G .

Remark 4.2. $G \subset \text{Iso}(\mathbb{R}_s^n)$ is unipotent, hence simply connected. Then Γ is finitely generated and torsion-free by [7, Theorem 2.10], as it is a discrete subgroup of G . Further, $\text{rk } \Gamma = \dim G$.

The fundamental theorem for finitely generated abelian groups states:

Lemma 4.3. *If Γ is abelian and torsion-free, then Γ is free abelian.*

By [4, Theorem 5.1.6], there exists a Malcev basis of G which generates Γ . We shall call it a *Malcev basis* of Γ .

Lemma 4.4. *Let $\gamma_1, \dots, \gamma_k$ denote a Malcev basis of Γ . If M is complete, then the translation parts v_1, \dots, v_k of the $\gamma_i = (I + A_i, v_i)$ are linearly independent.*

Proof. Γ has transitive centralizer in $\text{Iso}(\mathbb{R}_s^n)$. By continuity, so does G . Hence G acts freely on \mathbb{R}_s^n . Then the orbit map $G \rightarrow \mathbb{R}_s^n, g \mapsto g.0$, at the point 0 is a diffeomorphism onto the orbit $G.0$. Because G acts by affine transformations, $G.0$ is the span of the translation parts of the γ_i . So

$$k = \text{rk } \Gamma = \dim G = \dim G.0 = \dim \text{span}\{v_1, \dots, v_k\}.$$

So the v_i are linearly independent. □

4.1. Signature $(n-2, 2)$. As always, we assume $n-2 \geq 2$.

Proposition 4.5 (Wolf). *Let $M = \mathbb{R}_2^n/\Gamma$ be a flat pseudo-Riemannian homogeneous manifold. Then Γ is a free abelian group. In particular, the fundamental group of every flat pseudo-Riemannian homogeneous manifold M of dimension $\dim M \leq 5$ is free abelian.*

Proof. It follows from Corollary 3.11 that Γ has abelian holonomy. Consequently, if $\gamma = (I + A, v) \in \Gamma$ such that $A \neq 0$, then

$$A = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in a Witt basis with respect to U_Γ . Here, $C \neq 0$ is a skew-symmetric 2×2 -matrix, so we have $\text{rk } A = 2$. Because $\text{im } A \subset U_\Gamma$ and both these spaces are totally isotropic, we have $\dim \text{im } A = \dim U_\Gamma = 2$. Because $Av = 0$ we get $v \in \ker A = (\text{im } A)^\perp = U_\Gamma^\perp$. But then $Bv = 0$ for any $(I + B, w) \in \Gamma$, and as also $BA = 0$, it follows that $[(I + B, w), (I + A, v)] = (I + 2BA, 2Bv) = (I, 0)$. Hence Γ is abelian. It is free abelian by Lemma 4.3. \square

In the remainder of this section, the group Γ is always abelian, so the space $U_\Gamma = \sum_A \text{im } A$ is totally isotropic (in particular, $U_0 = U_\Gamma$). We fix a Witt decomposition with respect to U_Γ ,

$$\mathbb{R}_2^n = U_\Gamma \oplus W \oplus U_\Gamma^*$$

and any $v \in \mathbb{R}_2^n$ decomposes into $v = u + w + u^*$ with $u \in U_\Gamma$, $w \in W$, $u^* \in U_\Gamma^*$.

Remark 4.6. As seen in the proof of Proposition 4.5, if $\dim U_\Gamma = 2$, then $U_\Gamma = \text{im } A$ for any $\gamma = (I + A, v)$ with $A \neq 0$.

We can give a more precise description of the elements of Γ :

Proposition 4.7. *Let $M = \mathbb{R}_2^n / \Gamma$ be a complete flat pseudo-Riemannian homogeneous manifold. Then:*

- (1) Γ is generated by elements $\gamma_i = (I + A_i, v_i)$, $i = 1, \dots, k$, with linearly independent translation parts v_1, \dots, v_k .
- (2) If there exists $(I + A, v) \in \Gamma$ with $A \neq 0$, then in a Witt basis with respect to U_Γ ,

$$(4.1) \quad \gamma_i = (I + A_i, v_i) = \left(\begin{pmatrix} I_2 & 0 & C_i \\ 0 & I_{n-4} & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} u_i \\ w_i \\ 0 \end{pmatrix} \right)$$

with $C_i = \begin{pmatrix} 0 & c_i \\ -c_i & 0 \end{pmatrix}$, $c_i \in \mathbb{R}$, $u_i \in \mathbb{R}^2$, $w_i \in \mathbb{R}^{n-4}$.

- (3) $\sum_i \lambda_i w_i = 0$ implies $\sum_i \lambda_i C_i = 0$ (equivalently $\sum_i \lambda_i A_i = 0$) for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$.

Proof. We know from Proposition 4.5 that Γ is free abelian. Let $\gamma_1, \dots, \gamma_k$ denote a minimal set of generators with $\gamma_i = (I + A_i, v_i)$.

- (1) Lemma 4.4.
- (2) If $A \neq 0$ exists, then $U_\Gamma = \text{im } A$ is a 2-dimensional totally isotropic subspace. The matrix representation (4.1) is known from the proof of Proposition 4.5. As Γ is abelian, we have $A_i v_j = 0$ for all i, j . So $v_j \in \bigcap_i \ker A_i = U_\Gamma^\perp$ for all j .
- (3) Assume $\sum_i \lambda_i w_i = 0$ and set $C = \sum_i \lambda_i C_i$. Then $\sum_i \lambda_i (A_i, v_i) = (A, u)$, where $u \in U_\Gamma$. If $A \neq 0$, then G would have a fixed point (see Corollary 3.7). So $A = 0$, which implies $C = 0$. \square

Conversely, every group of the form described in the previous proposition defines a homogeneous space:

Proposition 4.8. *Let U be a 2-dimensional totally isotropic subspace of \mathbb{R}_2^n , and let $\Gamma \subset \text{Iso}(\mathbb{R}_2^n)$ be a subgroup generated by affine transformations $\gamma_1, \dots, \gamma_k$ of the form (4.1) with linearly independent translation parts. Further, assume that $\sum_i \lambda_i w_i = 0$ implies $\sum_i \lambda_i C_i = 0$ (equivalently $\sum_i \lambda_i A_i = 0$) for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. Then \mathbb{R}_2^n/Γ is a complete flat pseudo-Riemannian homogeneous manifold.*

Proof.

- (i) From the matrix form (4.1) it follows that Γ is free abelian, and the linear independence of the translation parts implies that it is a discrete subgroup of $\text{Iso}(\mathbb{R}_2^n)$.
- (ii) We check that the centralizer of Γ in $\text{Iso}(\mathbb{R}_2^n)$ acts transitively: Let $\mathfrak{iso}(\mathbb{R}_s^n)$ denote the Lie algebra of $\text{Iso}(\mathbb{R}_s^n)$. In the given Witt basis, the following are elements of $\mathfrak{iso}(\mathbb{R}_s^n)$:

$$S = \left(\begin{pmatrix} 0 & -B^\top & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right), \quad x, z \in \mathbb{R}^2, y \in \mathbb{R}^{n-2}.$$

Now assume arbitrary x, y, z are given. We will show that we can determine B such that S centralises $\log(\Gamma)$. Writing out the commutator equation $[S, (A_i, v_i)]$ blockwise, we see that $[S, (A_i, v_i)] = 0$ is equivalent to

$$-B^\top w_i = C_i z.$$

For simplicity, assume that w_1, \dots, w_j form a maximal linearly independent subset of w_1, \dots, w_k ($j \leq k$). As $-B^\top$ is a $2 \times (n-2)$ -matrix, the linear system

$$\begin{aligned} -B^\top w_1 &= C_1 z \\ &\vdots \\ -B^\top w_j &= C_j z \end{aligned}$$

consists of $2j$ linearly independent equations and $2(n-2)$ variables (the entries of B). As $\dim W = n-2 \geq j$, this system is always solvable.

So S can be determined such that it commutes with $\gamma_1, \dots, \gamma_j$. It remains to check that S also commutes with $\gamma_{j+1}, \dots, \gamma_k$. By assumption, each w_l ($l > j$) is a linear combination $w_l = \sum_{i=1}^j \lambda_i w_i$. Now $w_l - \sum_{i=1}^j \lambda_i w_i = 0$ implies $C_l - \sum_{i=1}^j \lambda_i C_i = 0$. But this means

$$-B^\top w_l = \sum_{i=1}^j \lambda_i \underbrace{(-B^\top w_i)}_{=C_i z} = \left(\sum_{i=1}^j \lambda_i C_i \right) z = C_l z,$$

so $[(A_l, v_l), S] = 0$.

The elements $\exp(S)$ generate a unipotent subgroup of the centralizer of Γ , so its open orbit at 0 is closed by [3, Proposition 4.10]. As x, y, z can be chosen arbitrarily, its tangent space at 0 is \mathbb{R}_2^n . Hence the orbit of the centralizer at 0 is open and closed, and therefore it is all of \mathbb{R}_2^n . Consequently, Γ has transitive centralizer.

- (iii) Because the centralizer is transitive, the action is free everywhere. It follows from [2, Proposition 7.2] that Γ acts properly discontinuously.

Now \mathbb{R}_2^n/Γ is a complete homogeneous manifold due to the transitive action of the centralizer on \mathbb{R}_2^n . \square

4.2. Dimension ≤ 5 .

Proposition 4.9 (Wolf). *Let $M = \mathbb{R}_s^n/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension ≤ 4 . Then Γ is a free abelian group consisting of pure translations.*

For a proof, see [8, Corollary 3.7.11].

Proposition 4.10. *Let $M = \mathbb{R}_s^5/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 5. Then Γ is a free abelian group. Depending on the signature of M , we have the following possibilities:*

- (1) Signature $(5, 0)$ or $(4, 1)$: Γ is a group of pure translations.
- (2) Signature $(3, 2)$: Γ is either a group of pure translations, or there exists $\gamma_1 = (I + A_1, v_1) \in \Gamma$ with $A_1 \neq 0$. In the latter case, $\text{rk } \Gamma \leq 3$, and if $\gamma_1, \dots, \gamma_k$ ($k = 1, 2, 3$) are generators of Γ , then v_1, \dots, v_k are linearly independent, and $w_i = \frac{c_i}{c_1} w_1$ in the notation of (4.1) ($i = 1, \dots, k$).

Proof. Γ is free abelian by Proposition 4.5. The statement for signatures $(5, 0)$ and $(4, 1)$ follows from Proposition 4.1.

Let the signature be $(3, 2)$ and assume Γ is not a group of pure translations. Then $U_\Gamma = \text{im } A$ is 2-dimensional (where $(I + A, v) \in \Gamma$, $A \neq 0$). By Lemma 4.4, the translation parts of the generators of Γ are linearly independent elements of U_Γ^\perp , which is 3-dimensional. So $\text{rk } \Gamma \leq 3$. Now, $U_\Gamma^\perp = U_\Gamma \oplus W$ with $\dim W = 1$. So the W -components of the translation parts are multiples of each other, and it follows from part (c) of Proposition 4.7 that $w_1 \neq 0$ and $w_i = \frac{c_i}{c_1} w_1$. \square

4.3. Dimension 6. In dimension 6, both abelian and non-abelian Γ exist.

We introduce the following notation: For $x \in \mathbb{R}^3$, let

$$T(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Then for any $y \in \mathbb{R}^3$,

$$T(x)y = x \times y,$$

where \times denotes the vector cross product on \mathbb{R}^3 .

Lemma 4.11. *Let $\Gamma \in \text{Iso}(\mathbb{R}_3^6)$ be a group with transitive centralizer in $\text{Iso}(\mathbb{R}_3^6)$. An element $X \in \log(\Gamma)$ has the form*

$$(4.2) \quad X = \left(\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u \\ u^* \end{pmatrix} \right)$$

with respect to the Witt decomposition $\mathbb{R}_3^6 = U_\Gamma \oplus U_\Gamma^*$. Furthermore,

$$C = \alpha_X T(u^*)$$

for some $\alpha_X \in \mathbb{R}$. If $[X_1, X_2] \neq 0$ for $X_1, X_2 \in \log(\Gamma)$, then $\alpha_{X_1} = \alpha_{X_2} \neq 0$.

Proof. The holonomy is abelian by Corollary 3.11, so (4.2) follows.

For $X \in \log(\Gamma)$ we have $Cu^* = 0$, that is, for any $\alpha \in \mathbb{R}$,

$$Cu^* = \alpha u^* \times u^* = 0.$$

If X is non-central, then $C \neq 0$ and $u^* \neq 0$. Now let $x, y \in \mathbb{R}^3$ such that u^*, x, y form a basis of \mathbb{R}^3 . Because C is skew,

$$u^{*\top} Cx = -u^{*\top} C^\top x = -(Cu^*)^\top x = 0.$$

Also,

$$x^\top Cx = -x^\top C^\top x \quad \text{and} \quad x^\top Cx = (x^\top Cx)^\top = x^\top C^\top x,$$

hence $x^\top Cx = 0$. So Cx is perpendicular to the span of x, u^* in the Euclidean sense¹. This means there is a $\alpha \in \mathbb{R}$ such that

$$Cx = \alpha u^* \times x.$$

In the same way we get $Cy = \beta u^* \times y$ for some $\beta \in \mathbb{R}$. As neither x nor y is in the kernel of C (which is spanned by u^*), $\alpha, \beta \neq 0$.

As y is not in the span of u^*, x , we have

$$\begin{aligned} 0 \neq x^\top Cy &= \beta x^\top (u^* \times y) \\ &= -y^\top Cx = -\alpha y^\top (u^* \times x) = -\alpha x^\top (y \times u^*) = \alpha x^\top (u^* \times y), \end{aligned}$$

where the last line uses standard identities for the vector product. So $\alpha = \beta$, and C and $\alpha T(u^*)$ coincide on a basis of \mathbb{R}^3 .

Now assume $[X_1, X_2] \neq 0$. Then

$$\alpha_2 u_2^* \times u_1^* = C_2 u_1^* = -C_1 u_2^* = -\alpha_1 u_1^* \times u_2^* = \alpha_1 u_2^* \times u_1^*,$$

and this expression is $\neq 0$ because $C_1 u_2^*$ is the translation part of $(\frac{1}{2}[X_1, X_2]) \neq 0$. So $\alpha_1 = \alpha_2$. \square

Proposition 4.12. *Let $M = \mathbb{R}_s^6/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume Γ is abelian. Then Γ is free abelian. Depending on the signature of M , we have the following possibilities:*

- (1) *Signature (6, 0) or (5, 1): Γ is a group of pure translations.*
- (2) *Signature (4, 2): Γ is either a group of pure translations, or Γ contains elements $\gamma = (I + A, v)$ with $A \neq 0$ subject to the constraints of Proposition 4.7. Further, $\text{rk } \Gamma \leq 4$.*
- (3) *Signature (3, 3): If $\dim U_\Gamma < 3$, then Γ is one of the groups that may appear for signature (4, 2). There is no abelian Γ with $\dim U_\Gamma = 3$.*

Proof. Γ is free abelian by Lemma 4.3. The statement for signatures (6, 0) and (5, 1) follows from Proposition 4.1.

If the signature is (4, 2) and Γ is not a group of pure translations, then the statement follows from Proposition 4.7. In this case, U_Γ^\perp contains the linearly independent translation parts and is of dimension 4. So $\text{rk } \Gamma \leq 4$.

Consider signature (3, 3). If $\dim U_\Gamma = 0$ or $= 2$, then Γ is a group as in the case for signature (4, 2). Otherwise, $\dim U_\Gamma = 3$. We show that in the latter case the centralizer of Γ does not act with open orbit: Any $\gamma \in \Gamma$ can be written as

$$\gamma = (I + A, v) = \left(\begin{pmatrix} I_3 & C \\ 0 & I_3 \end{pmatrix}, \begin{pmatrix} u \\ u^* \end{pmatrix} \right),$$

where $C \in \mathfrak{so}_3$ and $u, u^* \in \mathbb{R}^3$. In fact, we have $\mathbb{R}_3^6 = U_\Gamma \oplus U_\Gamma^*$ and $U_\Gamma^\perp = U_\Gamma$.

We will show that $u^* = 0$:

¹That is, with respect to the canonical positive definite inner product on \mathbb{R}^3 .

- (i) Because $\text{rk } C = 2$ for every $C \in \mathfrak{so}_3$, $C \neq 0$, but $U_\Gamma = \sum \text{im } A$ is 3-dimensional, there exist $\gamma_1, \gamma_2 \in \Gamma$ such that the skew matrices C_1 and C_2 are linearly independent. So, for every $u^* \in U_\Gamma^*$, there is an element $\gamma = (I + A, v)$ such that $Au^* \neq 0$.
- (ii) Γ abelian implies $A_1 u_2^* = 0$ for every $\gamma_1, \gamma_2 \in \Gamma$. With (i), this implies $u_2^* = 0$. So the translation part of every $\gamma = (I + A, v) \in \Gamma$ is an element $v = u \in U_\Gamma$.

Step (ii) implies $C_1 = \alpha_1 T(u_1^*) = 0$ by Lemma 4.11, but $C_1 \neq 0$ was required in step (i). Contradiction; so Γ cannot be abelian. \square

Proposition 4.13. *Let $M = \mathbb{R}_s^6/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume Γ is non-abelian. Then the signature of M is $(3, 3)$, and Γ is one of the following:*

- (1) $\Gamma = \Lambda \times \Theta$, where Λ is a discrete Heisenberg group and Θ a discrete group of pure translations in U_Γ . Then $3 \leq \text{rk } \Gamma = 3 + \text{rk } \Theta \leq 5$.
- (2) Γ is discrete group of rank 6 with center $Z(\Gamma) = [\Gamma, \Gamma]$ of rank 3. In this case, M is compact.

Proof. If the signature was anything but $(3, 3)$ or $\dim U_0 < 3$, then Γ would have to be abelian due Proposition 4.5. The holonomy is abelian by Corollary 3.11.

For the following it is more convenient to work with the real Zariski closure G of Γ and its Lie algebra \mathfrak{g} . As \mathfrak{g} is 2-step nilpotent, $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}(\mathfrak{g})$, where \mathfrak{v} is a vector subspace of \mathfrak{g} of dimension ≥ 2 spanned by non-central elements. Set $\mathfrak{v}_\Gamma = \mathfrak{v} \cap \log(\Gamma)$. We proceed in four steps:

- (i) Assume there are $X_i = (A_i, v_i) \in \mathfrak{v}$, $\lambda_i \in \mathbb{R}$, $v_i = u_i + u_i^*$ (for $i = 1, \dots, m$), such that $\sum_i \lambda_i u_i^* = 0$. Then $\sum_i \lambda_i X_i = (\sum_i \lambda_i A_i, \sum_i \lambda_i v_i) = (A, u) \in \mathfrak{v}$, where $u \in U_\Gamma$. For all $(A', v') \in \mathfrak{g}$, the commutator with (A, u) is $[(A', v'), (A, u)] = (0, 2A'u) = (0, 0)$. Thus $(A, u) \in \mathfrak{v} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$.
So if $X_1, \dots, X_m \in \mathfrak{v}$ are linearly independent, then $u_1^*, \dots, u_m^* \in U_0^*$ are linearly independent (and by Lemma 4.11 the C_1, \dots, C_m are too). But $\dim U_0^* = 3$, so $\dim \mathfrak{v} \leq 3$.
- (ii) If $Z \in \mathfrak{z}(\mathfrak{g})$, then $C_Z = 0$ and $u_Z^* = 0$: As Z commutes with X_1, X_2 , we have $C_Z u_1^* = 0 = C_Z u_2^*$. By step (i), u_1^*, u_2^* are linearly independent. So $\text{rk } C_Z < 2$, which implies $C_Z = 0$ because C_Z is a skew 3×3 -matrix. Also, $C_1 u_Z^* = 0 = C_2 u_Z^*$, so $u_Z^* = \ker C_1 \cap \ker C_2 = \{0\}$. So $\exp(Z) = (I, u_Z)$ is a translation by $u_Z \in \Gamma$.
- (iii) Assume $\dim \mathfrak{v} = 2$. Let \mathfrak{v} be spanned by X_1, X_2 , and $Z_{12} = [X_1, X_2]$ is a pure translation by an element of U_Γ . The elements X_1, X_2, Z_{12} span a Heisenberg algebra \mathfrak{h}_3 contained in \mathfrak{g} . If $\dim \mathfrak{g} > 3$, then $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z_{12} \oplus \mathfrak{t}$, where according to step (ii) \mathfrak{t} is a subalgebra of pure translations by elements of U_Γ . So $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{t}$ with $0 \leq \dim \mathfrak{t} < \dim U_\Gamma = 3$. This gives part (a) of the proposition.
- (iv) Now assume $\dim \mathfrak{v} = 3$. We show that $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{v}, \mathfrak{v}]$ and $\dim \mathfrak{z}(\mathfrak{g}) = 3$: Let $X_1 = (A_1, v_1), X_2 = (A_2, v_2) \in \mathfrak{v}_\Gamma$ such that $[X_1, X_2] \neq 0$. By Lemma 4.11, $C_1 = \alpha T(u_1^*)$ and $C_2 = \alpha T(u_2^*)$ for some number $\alpha \neq 0$. There exists $X_3 \in \mathfrak{v}_\Gamma$ such that X_1, X_2, X_3 form basis of \mathfrak{v} . By step (i), u_1^*, u_2^*, u_3^* are linearly independent. For $i = 1, 2$, $\ker C_i = \mathbb{R}u_i^*$, and u_3^* is proportional to neither u_1^* nor u_2^* . This means $C_1 u_3^* \neq 0 \neq C_2 u_3^*$, which implies $[X_1, X_3] \neq 0 \neq [X_2, X_3]$. By Lemma 4.11, $C_3 = \alpha T(u_3^*)$.

Write $Z_{ij} = [X_i, X_j]$. The non-zero entries of the translation parts of the commutators Z_{12} , Z_{13} and Z_{23} are

$$C_1 u_2^* = \alpha u_1^* \times u_2^*, \quad C_1 u_3^* = \alpha u_1^* \times u_3^*, \quad C_2 u_3^* = \alpha u_2^* \times u_3^*.$$

Linear independence of u_1^*, u_2^*, u_3^* implies that these are linearly independent. Hence the commutators Z_{12}, Z_{13}, Z_{23} are linearly independent in $\mathfrak{z}(\mathfrak{g})$. Because $\dim \mathfrak{g} = \dim \mathfrak{v} + \dim \mathfrak{z}(\mathfrak{g}) \leq 6$, it follows that $\mathfrak{z}(\mathfrak{g})$ is spanned by these Z_{ij} , that is $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{v}, \mathfrak{v}]$. This gives part (b) of the proposition. \square

Remark 4.14. In case (2) of Proposition 4.13 it can be shown that Γ is a lattice in a Lie group $H_3 \ltimes_{\text{Ad}^*} \mathfrak{h}_3^*$, see [5, Section 5.3].

We have a converse statement to Proposition 4.13:

Proposition 4.15. *Let Γ be a subgroup of $\text{Iso}(\mathbb{R}_3^6)$. Then $M = \mathbb{R}_3^6/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold if there exists a 3-dimensional totally isotropic subspace U and Γ is a group of type (1) or (2) in Proposition 4.13 (with U_Γ replaced by U).*

Proof. Both cases can be treated simultaneously.

Let $X_1, X_2, X_3 \in \log(\Gamma)$ such that the $\exp(X_i)$ generate Γ . The number $\alpha \neq 0$ from Lemma 4.11 is necessarily the same for X_1, X_2, X_3 .

- (i) The group Γ is discrete because the translation parts of the generators $\exp(X_i)$ and those of the generators of $Z(\Gamma)$ form a linearly independent set.
- (ii) We show that the centralizer of Γ is transitive. Consider the following elements

$$S = \left(\begin{pmatrix} 0 & -\alpha T(z) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \in \mathfrak{iso}(\mathbb{R}_3^6)$$

with $x, z \in \mathbb{R}^3$ arbitrary. Then $[X_i, S] = 0$ for $i = 1, 2, 3$, because

$$C_i z = \alpha u_i^* \times z = -\alpha z \times u_i^* = -\alpha T(z) u_i^*.$$

Clearly, S also commutes with any translation by a vector from U . So in both cases (1) and (2), Γ has a centralizer with an open orbit at 0. The exponentials of the elements of S clearly generate a unipotent subgroup of $\text{Iso}(\mathbb{R}_3^6)$, hence the open orbit is also closed and thus all of \mathbb{R}_3^6 .

- (iii) From the transitivity of the centralizer, it also follows that the action is free and thus properly discontinuous ([2, Proposition 7.2]).

So \mathbb{R}_3^6/Γ is a complete homogeneous manifold. \square

In the situation of Proposition 4.13 it is natural to ask whether the statement can be simplified by claiming that Γ is always a subgroup of a group of type (2) in Proposition 4.13. But this is not always the case:

Example 4.16. We choose the generators $\gamma_i = (I + A_i, v_i)$, $i = 1, 2$, of a discrete Heisenberg group Λ as follows: If we decompose $v_i = u_i + u_i^*$ where $u_i \in U_\Gamma$, $u_i^* \in U_\Gamma^*$, let $u_i = 0$, $u_i^* = e_i^*$, $\alpha = 1$ (with α as in the proof of Proposition 4.13 and e_i^* refers to the i th unit vector taken as an element of U_Γ^*). Then $\gamma_3 = [\gamma_1, \gamma_2] = (I, v_3)$, where $u_3 = e_3$, $u_3^* = 0$. Let $\gamma_4 = (I, u_4)$ be the translation by $u_4 = \sqrt{2}e_1 + \sqrt{3}e_2 \in U_\Gamma$. Let $\Theta = \langle \gamma_4 \rangle$ and $\Gamma = \Lambda \cdot \Theta (\cong \Lambda \times \Theta)$.

Assume there exists $X = (A, v)$ of the form (4.2) not commuting with X_1, X_2 . Then the respective translation parts of $[X_1, X]$ and $[X_2, X]$ are

$$e_1 \times u^* = \begin{pmatrix} 0 \\ -\eta_3 \\ \eta_2 \end{pmatrix}, e_2 \times u^* = \begin{pmatrix} \eta_3 \\ 0 \\ -\eta_1 \end{pmatrix} \in U_\Gamma$$

where η_i are the components of u^* , and $\eta_3 \neq 0$ due to the fact that X and the X_i do not commute. If Γ could be embedded into a group of type (2), such X would have to exist. But by construction u_4 is not contained in the \mathbb{Z} -span of $e_3, e_1 \times u^*, e_2 \times u^*$. So the group generated by Γ and $\exp(X)$ is not discrete in $\text{Iso}(\mathbb{R}_3^6)$.

5. FUNDAMENTAL GROUPS OF COMPLETE FLAT PSEUDO-RIEMANNIAN HOMOGENEOUS SPACES

In this section we will prove the following:

Theorem 5.1. *Let Γ be a finitely generated torsion-free 2-step nilpotent group of rank n . Then there exists a faithful representation $\varrho : \Gamma \rightarrow \text{Iso}(\mathbb{R}_n^{2n})$ such that $M = \mathbb{R}_n^{2n}/\varrho(\Gamma)$ a complete flat pseudo-Riemannian homogeneous manifold M of signature (n, n) with abelian linear holonomy group.*

We start with a construction given in [1, Paragraph 5.3.2] to obtain nilpotent Lie groups with flat bi-invariant metrics. Let \mathfrak{g} be a real 2-step nilpotent Lie algebra of finite dimension n . Then the semidirect sum $\mathfrak{h} = \mathfrak{g} \oplus_{\text{ad}^*} \mathfrak{g}^*$ is a 2-step nilpotent Lie algebra with Lie product

$$(5.1) \quad [(X, \xi), (Y, \eta)] = ([X, Y], \text{ad}^*(X)\eta - \text{ad}^*(Y)\xi),$$

where $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$ and ad^* denotes the coadjoint representation. An invariant inner product on \mathfrak{h} is given by

$$(5.2) \quad \langle (X, \xi), (Y, \eta) \rangle = \xi(Y) + \eta(X).$$

Its signature is (n, n) , as the subspaces \mathfrak{g} and \mathfrak{g}^* are totally isotropic and dual to each other.

If G is a simply connected 2-step nilpotent Lie group with Lie algebra \mathfrak{g} , then $H = G \ltimes_{\text{Ad}^*} \mathfrak{g}^*$ (with \mathfrak{g}^* taken as a vector group) is a simply connected 2-step nilpotent Lie group with Lie algebra \mathfrak{h} , and $\langle \cdot, \cdot \rangle$ induces a bi-invariant flat pseudo-Riemannian metric on H .

Remark 5.2. For any lattice $\Gamma_H \subset H$, the space H/Γ_H is a compact flat pseudo-Riemannian homogeneous manifold. In particular, H is complete (see [6, Proposition 9.39]). By [2, Theorem 3.1], Γ_H has abelian linear holonomy.

Proof of Theorem 5.1. Let Γ be a finitely generated torsion-free 2-step nilpotent group. The real Malcev hull G of Γ is a 2-step nilpotent simply connected Lie group such that Γ is a lattice in G . In particular, $\text{rk } \Gamma = \dim G = n$. If \mathfrak{g} is the Lie algebra of G , let H be as in the construction above. We identify G with the closed subgroup $G \times \{0\}$ of H . As Γ is a discrete subgroup of H , it follows from the remark above that $M = H/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold with abelian linear holonomy.

As H has signature (n, n) , the development representation ϱ of the right-multiplication of G gives the representation of Γ as isometries of \mathbb{R}_n^{2n} . \square

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REFERENCES

- [1] O. Baues, *Flat pseudo-Riemannian manifolds and prehomogeneous affine representations*, in 'Handbook of Pseudo-Riemannian Geometry and Supersymmetry', EMS, IRMA Lect. Math. Theor. Phys. 16, 2010, pp. 731-817 (also arXiv:0809.0824v1)
- [2] O. Baues, W. Globke, *Flat Pseudo-Riemannian Homogeneous Spaces With Non-Abelian Holonomy Group*, Proc. Amer. Math. Soc. 140, 2012, pp. 2479-2488 (also arXiv:1009.3383)
- [3] A. Borel, *Linear Algebraic Groups*, 2nd edition, Springer, 1991
- [4] L. Corwin, F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications*, Cambridge University Press, 1990
- [5] W. Globke, *Holonomy Groups of Flat Pseudo-Riemannian Homogeneous Manifolds*, Dissertation, Karlsruhe Institute of Technology, 2011
- [6] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, 1983
- [7] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, 1972
- [8] J.A. Wolf, *Spaces of Constant Curvature*, 6th edition, Amer. Math. Soc., 2011
- [9] J.A. Wolf, *Homogeneous manifolds of zero curvature*, Trans. Amer. Math. Soc. 104, 1962, pp. 462-469

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